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Research Reports

A – G

FROM INTERPRETATIVE KNOWLEDGE TO SEMIOTIC INTERPRETATIVE KNOWLEDGE IN PROSPECTIVE TEACHERS' FEEDBACK TO STUDENTS' SOLUTIONS

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This study broadens the notion of Interpretative Knowledge (IK) into Semiotic Interpretative Knowledge (SIK) by considering the role of semiotic systems in mathematical thinking and learning. We analyse the relationship between SIK and feedback structured in four categories according to the semiotic functions involved. Based on quantitative and qualitative data we scrutinize the SIK and feedback deployed by prospective primary school teachers. Although connected with teachers' mathematical knowledge, SIK is a specialized knowledge that requires specific training.

INTRODUCTION

Starting from research related to the conceptualization of mathematical knowledge for teaching (MKT) (Ball et al., 2008), Ribeiro and co-authors (2013) introduce the notion of interpretative knowledge (IK) as the part of the mathematical knowledge "that allows teachers to give sense to pupils' non-standard answers (i.e., adequate answers that differ from the ones teachers would give or expect) or to answers containing errors" (Ribeiro et al., 2016, p. 9). Being IK a kind of knowledge related to problem-solving strategies and errors, it is a piece of typically conceptual or strategic knowledge and does not consider explicitly the semiotic aspects related to signs and sign use in mathematical activity. As highlighted by Duval, conceptualization cannot be accomplished without an adequate competence of what Ernest (2006) calls patterns of sign use and production. As research shows (e.g., Duval, 2017; Ferretti et al., 2022), interpreting students behavior requires a strong semiotic competence.

In Asenova et al. (2023) the notion of semiotic interpretative knowledge (SIK) is introduced, broadening the seminal notion of IK proposed in the literature (Mellone et al., 2020; Ribeiro et al., 2013, 2016). This study undergoes a further step analysing prospective teachers' spontaneous use of SIK in interpreting students' responses and in providing feedback. In this sense, the approach on feedback based on the development of a suitable IK, proposed by Galleguillos & Ribeiro (2019), is extended by considerations related to SIK with the aim to show that SIK represents a theoretical tool able to further deepen the nature of teachers' feedback.

THEORETICAL FRAMEWORK

Ball and co-authors (2008) introduced the construct of mathematical content for teaching (MKT) as the mathematical knowledge needed by teachers to perform the usual tasks related to teaching mathematics. MKT is made of subject matter knowledge

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(SMK), related to the specificities of mathematics, and pedagogical content knowledge (PCK), related to the specificities of teaching and learning of mathematics. Two subcategories of SMK are common content knowledge (CCK) and of specialized content knowledge (SCK), while two subcategories of PCK are knowledge of content and students (KCS) and knowledge of content and teaching (KCT). While CCK is a MKT independent of the teaching-learning context, SCK is a SMK specific to it. Rooted in Ball and colleagues' notion of MKT, Ribeiro et al. (2013) introduced the construct of interpretative knowledge (IK) as a kind of SMK "in the intersection between the common content knowledge and the specialized content knowledge" (p. 4). Di Martino et al. (2019) derive the characterization of IK as belonging to SCK, but as strongly related to the CCK, from the conclusion that a strong CCK is necessary but not sufficient to develop a good level of IK, but at the same time, teachers with a strong CCK have difficulties in accepting unusual strategies that differ from their own (Asenova, 2022). Beside the conceptual, strategic and affective aspects (Di Martino et al., 2016) investigated in research on IK, the semiotic aspects of IK are still little explored. A strong semiotic competence is indispensable for a cognitively meaningful mathematical activity, but semiotic is not a MKT, as for instance geometry or algebra. It might be for this reason that IK does not consider the intrinsically semiotic nature of mathematical cognitive functioning. According to Duval (2017), in mathematics, ostensive references are impossible, as we cannot directly access mathematical objects through our senses. We can say that conceptualisation itself, in mathematics, is identified with this complex coordination of several semiotic systems (Duval, 2017; Ernest, 2006), rooted in semiotic transformations within the same semiotic system (treatments) and semiotic transformations between different semiotic systems (conversions). D'Amore (2003) identifies conceptualisation with the following semiotic functions, specific to mathematics: (1) choice of the distinctive features of a mathematical object; (2) treatment in the same semiotic system; (3) conversion between semiotic systems. The management of such semiotic complexity, within the structure of semiotic systems and the processing of semiotic functions, comes up against Duval's famous cognitive paradox (Duval, 2017): On the one hand we know abstract mathematical objects only through the semiotic activity mentioned above; on the other hand, such a semiotic activity requires the conceptual knowledge of the mathematical objects on the part of the student.

Taking into account the intrinsically semiotic nature of mathematical thinking, in Asenova et al. (2023) the theoretical construct of SIK is introduced as "the knowledge needed by teachers in order to interpret students' answers (be they standard or non-standard), as well as students' behavior, and to give an appropriate feedback to them, when conceptual knowledge is hindered, and thus remains hidden behind difficulties related to patterns of sign use and production, including individual creativity in sign use" (p. 11). SIK lies at the crossover of SMK and PCK because the control of semiotic functions is intertwined both with mathematical knowledge (noesis and semiosis are overlapped) and their implementation in the teaching-learning activity driven by the teacher (KCS, KCT). We show that besides a mere conceptual IK, in the context of students with special educational needs, a strong SIK seems to be necessary to provide effective feedback.

Feedback is defined by Hattie and Timperly (2007) as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (p. 81). These authors distinguish, among others, between feedback about the task (FT) and feedback about the processing of the task (FP). FT and FP can be more or less elaborated and can go from simple correct/incorrect information to more constructive feedback related to additional information on content and strategies to be used. In teacher training, it is important to provide prospective teachers with the skills needed for a wide range of possible feedbacks and to foster the IK needed to give FT. Galegiullos and Ribeiro (2019) investigate prospective teachers' ability to use IK in giving FT: Teachers were asked to work in groups and first solve a task and then provide feedback to some solutions given by students to the same task. These authors classify the provided feedback into four categories: (a) Feedback on how to solve the problem; (b) Confusing feedback: When the feedback seems to be correct, but it can be confusing for the student; (c) Counterexample as feedback; (d) Superficial feedback: The content of such feedback was insufficient (too broad or inconsistent) to allow the solver to understand its meaning. In this paper we develop the kinds of feedback, introduced by Galegiullos and Ribeiro, consistently with the notion of SIK; FT is elaborated according to the implementation of the semiotic functions. We have outlined four main categories of feedback: type (i) - no mention of semiotic functions, which is framed by Galegiullos and Ribeiro's categories; type (ii) - use of semiotic representations confined to the recognition of the distinctive traits; type (iii) - use of distinctive traits and treatments; type (iv) - use of distinctive traits, treatments, and conversions. The semiotic categorization of feedback does not provide a level of effectiveness *per se* but filling the gap "between what is understood and what is aimed to be understood" (Hattie & Timperley, 2007, p. 82) is specific to the context. Feedback is intertwined with the nature of the task, the student's specificities, and the learning environment. In this paper, the accomplishment of SIK on the part of prospective teachers and its benefit for unfolding their feedback effectiveness is investigated, showing how it can be used to further deepen the understanding on the relations between prospective teachers' (S)IK and their ability to provide FT.

METHODOLOGY OF RESEARCH

In our investigation, we first focus on the classification and analysis of the answers given by 180 primary school prospective mathematics teachers (PPTs) attending an Italian University to a questionnaire related to the interpretation of the incorrect answers given by students to four math tasks and to the feedback they would give to the student. Then we focus on the way PPTs use SIK to support their answers to the questionnaire in a follow up interview. We use the PPT's answers to the questionnaire to classify the IK used to interpret the student's answer for themselves and the IK used to provide FT, adopting the following categories: (0) does not respond or the answer is not classifiable; (1) Conceptual IK: The PPT does not mention representations but only concepts, strategies; (2) SIK-R: The PPT mention only semiotic representations without reference to semiotic transformations; (3) SIK-T: The PPT refers to treatments in the same semiotic system; (4) SIK-C: The PPT refers to conversions between different systems.

The 180 PPTs were in the first year of the 10-semester master's degree-course in primary education and 21 gave their permission to be interviewed after completing the questionnaire. In the following, we discuss and analyse the interview of a PPT.

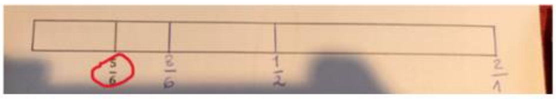
In elaborating the tasks we took our cue from the methodology used in Ribeiro et al. (2013) but instead of asking the PPTs to first solve the problem and then to give feedback to student's solutions, we presented immediately the student's solution and then asked the PPTs to first interpret the solution and then to provide feedback. We chose this approach because our focus was not on the development of the teachers' IK, but on the kind of IK used spontaneously by the PPTs in giving feedback.

Here we focus on the first two tasks of the questionnaire (Figure 1 and 2).

Figure 1: Task 1 (with the kind permission of prof. Cristina Sabena, inspired by prof. Elisabetta Robotti's research on teaching-learning of fractions)

Task 1

Starting from the representation of the number $\frac{5}{6}$, Gina have to represent the numbers $\frac{8}{6}$, $\frac{1}{2}$ and 2.



Question 1.1: What do you think has happened? **Question 1.2**: How would you intervene?

Figure 2: Task 2 (used by the authors in teacher training courses)

Task 2

Giovanni likes to invest his weekly tip in the stock exchange. He bought a share of Tetris, his favourite video game. In the table Giovanni represented the share's performance in the first three months of the investment he started in January.

Gennaio	7 €	
Febbraio	9,1 €	+30%
Marzo	7,7 €	+10%

Milena, after looking at Giovanni's table, claims that the share value of Tetris has decreased by 20 per cent from February to March.

Question 2.1: What do you think has happened in Milena's interpretation?

Question 2.2: How would you intervene?

Task 1 was chosen because it drives the use of semiotic functions (conversions involving symbolic language, natural language and figural representations). Task 2 was chosen because the implementation of the network of semiotic functions is not immediate and our conjecture is that the students would focus more on conceptual IK and treatments confined to algorithms (calculation and confront of percentages).

RESULTS AND DISCUSSION

Regarding Task 1, the quantitative data shows that most of the PPTs opt for IK both in the interpretation of the solution (52,2%) and in the feedback (34,4%). There was a high percentage of invalid answers to Question 1.1 (22,2 %) and Question 1.2 (33,3%) that can be traced back to a lack of CCK and SCK that hinders IK and the ensuing feedback. A high percentage (22%) of the PPTs provide type (iv) feedback based on the implementation of the three semiotic functions even if only almost a third (7,8%) display SIK-C in the interpretation of the data. This case testifies that when PPTs possess suitable semiotic competences, which lie at the crossover of SCK and PCK, they set out SIK-C and feel the need to ground their feedback in the networking of different semiotic systems for higher effectiveness and clarity.

In Task 2, we notice a prevalence of invalid interpretations and feedback (Question 2.1: 40%, Question 2.2: 68,9%). There is a high percentage of PPTs who resort to IK in the interpretation (33%) but a lower percentage of PPTs who are able to give type (i) feedback (20,5%). A significative percentage (26,7%) of PPTs resorted to SIK-T for the interpretation but only 10,5% provided a type (iii) feedback. To make sense of this result we must consider that the task was challenging for the PPTs in that it was difficult for them to unravel the mathematical knowledge (percentages) in terms of CCK and SCK, and the ensuing KCS and KCT. Thus, on the one hand the IK was not backed by CCK and SCK to carry out an appropriate interpretation of the solution and provide effective feedback. On the other hand, it was difficult for the PPTs displaying semiotic activity at the crossover of SCK and PCK in the interpretation and feedback. Indeed, the SIK-T does not amount to a true semiotic interpretation but to meaningless calculations carried out in symbolic language. They testify the identification of the mathematical object with the semiotic representations accountable to the cognitive paradox.

Sara's Interview

In order to operationalize our theoretical lens for interpretating tasks and providing feedback, we present an excerpt from the interview of Sara concerning Task 1. Sara interprets the solution with SIK-T explaining that the solution does not correctly consider the meaning of the denominators for the ordering of the fractions. She provides a type (iii) feedback based on the ordering of the fractions in the arithmetic symbolic system. The researcher asks Sara to explain her feedback and to make her feedback more effective. She spontaneously performs treatments and conversions that also involve Montessori materials she uses at school with her students. After transforming via treatment all the fractions to 6 as common denominator she grabs the Montessori-rod (Figure 3a).

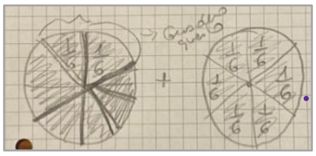
Sara: If I want to represent $\frac{1}{2}$ which is halfway the length of the rod, three coloured rectangles on one side and three on the other. I consider three of the coloured rectangles [she scrolls the small red rectangle over the rod and counts 1, 2, 3 (Figure 3b)]. The same holds for 5/6 [she scrolls the red rectangle counting 1, 2, 3, 4, 5].

Researcher: How would you represent $\frac{8}{6}$ with Montessori-rods?

Sara: The only thing I can think of is to take … [she grabs a rod with 6 colored rectangles and a rod with 2 colored rectangles (Figure 3c)]. One represents the number 6 and the other represents the number 2, the 6-rod represents quantity 6 and the 2-rod represents quantity 2. I jump in quantity because I start with the number 1, this red one. When I count I do so 1, 2, 3 ... [scrolls the 1-rod over the 6-rod counting 1, 2, 3, 4, 5]. This is the useful step to take, this is the 6-rod but I consider 1, 2, 3, 4, 5 [referring to $\frac{5}{6}$]. If I consider this [the 2-rectangles colored rod] it means I consider 8 parts, it means I consider 8 units and I'm on the wrong path, I'm somewhere else like this. I no longer have the base and that's it, but I have the base plus two and it comes 8 as a whole. I mess up the student's understanding. So to do $\frac{8}{6}$ it's easier to use pie charts, where I consider all the quantities. The half divides me into two parts, because these wedges are made equal. I have the $\frac{1}{2}$ and $\frac{1}{2}$ and it gives me the whole, then I have the $\frac{2}{4}$, $\frac{3}{4}$ and so on. It's all divided in this fashion [drawing on a sheet of paper (Figure 4)] 1, 2, 3, 4, [counting the wedges on the pie chart] up to 6. So, I take another pie chart divided in six wedges and I consider this and this and this 8 times [writing $\frac{1}{6}$ on each of the 8 slices she is pointing to (Figure 4)].

Figure 3: The Montessori-rods used by Sara to represent the fractions

Figure 4: Sara's drawing of the pie charts used to represent the fractions



The protocol shows Sara possesses SIK-T that allows her to interpret the solution and provide basic type (iii) feedback. When prompted by the researcher to explain the feedback she would share with the students she feels the need to include other semiotic systems via conversion transformations. Sara is aware that tapping into a network of semiotic transformations, which involve more semiotic systems, empowers the efficacy of the feedback she can provide. Nevertheless, Sara's interview highlights the lack of appropriate SIK that lies at the crossover of CCK, SCK and KCS and KCT. In fact, SIK as a specific MKT requires solid subject content knowledge (CCK and SCK) combined with PCK in order to implement patterns of sign use and production coherent with the mathematics at stake an effective to the student's learning. When Sara crosses the borders of treatment transformations in the arithmetic language to include Montessori- rods and pie charts, she loses control of the meaning of fractions and undermines the efficacy of her type (iv) feedback providing confusing information. Indeed, she mixes the meaning of the colored rectangles of the rods, without recognizing the distinctive traits of the semiotic representation. On the one hand the rectangles represent the unitary fraction $\frac{1}{6}$, on the other hand they represent an integer quantity, the number of parts in which the whole is divided, 6 parts in the rod with six rectangles. When she wants to represent $\frac{8}{6}$, she is puzzled because “if I consider this (the 2-rectangles colored rod) it means I consider 8 parts, it means I consider 8 units and I’m on the wrong path”. She means that she is not considering $\frac{1}{6}$ as unitary fraction but $\frac{1}{8}$. With the pie charts she recollects the correct meaning of the fraction because the pie has no fixed unit. She claims that: “The half divides me into two parts, because these graphs are made equal. I have the $\frac{1}{2}$ and $\frac{1}{2}$ and it gives me the whole, then I have the $\frac{2}{4}$, $\frac{3}{4}$ and so on. It’s all divided in this fashion.” So, each pie is the whole divided in 6 parts and each part is $\frac{1}{6}$. She then considers 8 parts that represent $\frac{8}{6}$. Sara has not yet reached an appropriate competence in semiotics that allows her to handle conversions in the context of fractions, thus developing an appropriate SIK.

CONCLUSIONS

The aim of the research was to investigate the spontaneous use of SIK by prospective primary school teachers, prior to specific training in mathematics specialized knowledge. We also analysed the impact of SIK on feedback categorized according to semiotic parameters. The quantitative data show that SIK does not belong to prospective teachers as a consequence of their subject matter knowledge. When prospective teachers recur to SIK they deploy the network of semiotic functions especially when providing feedback. Their need to provide effective information is characterized by a type (iv) feedback. Although SIK is not a spontaneous consequence of subject matter knowledge, we can infer that it is a necessary condition to trigger SIK because “there is no noesis without semiosis” (Duval, 2017, p. 23). Sara’s protocol shows a spontaneous use of SIK-T both in the interpretation and the feedback but her need for further clarity and efficacy is backed by type (iv) feedback that requires the interplay of all the semiotic functions. Sara’s type (iv) feedback clashes against the lack of semiotic competences that would allow her to position her SIK at the crossover of SMK and PCK. We can conclude that SIK is an important instrument in the hands of the teacher to interpret students’ behavior and give effective feedback. The interiorization of SIK requires specific training in prospective teachers’ professional development. Further research is required to design appropriate training programs that include SIK and validate its effectiveness in providing feedback able to improve students’ mathematical learning.

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